Teachers' professional practice conducting mathematical discussions [1]

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Abstract. This paper seeks to identify actions that can be regarded as building elements of teachers' classroom practice in mathematical discussion and how these actions may be combined to provide fruitful learning opportunities for students. It stands on a framework that focuses on two key elements of teaching practice: the tasks that teachers propose to students and the way teachers handle classroom communication. Tasks are appraised concerning their level of challenge. Teachers' actions in discussions are classified as informing/suggesting, guiding, and challenging. The methodology is qualitative with data collected from video recording of the classroom. The analysis of classroom episodes dealing with rational numbers but with different agendas, such as providing students opportunities for learning about representations, concepts, connections, and procedures and for developing reasoning suggests that some degree of challenge promotes fruitful learning situations. However, such situations tend to require preparation and follow-up with guiding and even informing/suggesting actions so that the students can learn what has been set in the teacher's agenda.

Key words: Teacher practice; Classroom communication; Challenge; Rational numbers.

Introduction

Since the 1990s, teachers' professional practice has become a pervasive issue in research in mathematics education (Ponte & Chapman, 2006). However, in many studies, the concept of teaching practice is not discussed in depth. In this article, we aim to contribute to the understanding of this notion, by describing and illustrating a model to analyse teaching practice that put special attention to the tasks proposed to the students and to the actions taken in handling classroom communication.

Mathematics teachers' practices may be quite distinct in different school levels, with students of different socioeconomic status and ability, in teaching different mathematics topics, and in different curriculum frameworks. We choose to pursue our aim of studying teaching practices at the end of primary school (fifth grade), in teaching rational numbers, in the framework of an exploratory approach (Ponte, 2005; Ruthven, 1989), in moments of whole class discussions. To embrace the wide complexity of the teachers' activity, we seek to combine a view that looks at teachers' actions at a micro level taking into account the teachers' aims at a broader level. Our specific aim is to indicate, in the setting that we have chosen to study, the actions that may be identified on the teacher's classroom activity during mathematical discussions, and how they may be combined to provide fruitful learning opportunities for students.

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Conceptualizing teaching practice

Practices may be seen as socially organized activities recurrently undertaken, taking into account the meanings that participants assign to what they do (Ponte & Chapman, 2006). The nature of the activity is related to the teacher's motives, which originate aims to be achieved through various teaching actions (Christiansen & Walther, 1986). To understand an activity, we need to identify the problems and opportunities seized by the person during its realization that may lead to decisions regarding keeping or changing the original action plans (Schoenfeld, 2010).

Mathematics teachers' classroom professional practices may be seen as framed by two basic elements: the tasks proposed to the students, and the communication processes that take place in the classroom. The tasks that the teacher proposes in the mathematics classroom are key starting points for students' learning (NCTM, 1991). In routine tasks, students need to provide an immediate response or to apply a solution method that they already know, and they just have to identify, remember, and carry out such method. Challenging tasks, on the other hand, allow a variety of strategies that may be compared and evaluated, giving room for interesting discussions. In many classrooms, the most common task is the exercise, which aims that students apply an already learned solution method. This kind of task has long been recognized as pervasive in mathematics classrooms (Christiansen & Walther, 1986). More challenging tasks such as problems, projects, investigations and explorations may be useful for mathematics classrooms (Ponte, 2005). For students, it makes a great difference dealing with tasks just applying knowledge they have already learned and with tasks that require an effort to understand and formulate a new solving strategy. The critical importance of using worthwhile mathematical tasks to support students' learning is a central idea in mathematics education (NCTM, 1991). Tasks may have different structure (open/closed) and degree of complexity (high/low), involve different contexts (real-life/mathematical and familiar/unfamiliar), presentation mode, and time and setting for doing them (Ponte, 2005). Furthermore, Stein, Remillard & Smith (2007) note that, sometimes, teachers propose task at a high level of cognitive demand but their subsequent interventions lead to a lowering of this level and of the tasks' value for learning.

Another aspect that frames teachers' professional practices is the communication that is established in the classroom (Bishop & Goffree, 1986; Franke, Kazemi, & Battey, 2007). In univocal communication, there is a voice that prevails over all others, whereas, in dialogic communication, many partners participate in a similar way (Brendefur & Frykholm, 2000). In many classes, univocal communication clearly predominates. Ruthven, Hofmann and Mercer (2011) consider that dialogic communication is possible in teaching situations when the teacher "takes different points of view seriously . . . Encouraging students to talk in an exploratory way that supports the development of understanding" (p. 4-81). Educational research has long pointed out a very common type of communication in teaching contexts, the I-R-E (initiation-response-evaluation) triadic sequence (Franke, Kazemi, & Battey, 2007). The teacher begins by asking a question (initiation), which is followed by a response from a student, which, in turn, prompts the teacher's evaluation. This kind of communication leaves little room for students' creative participation.

A fundamental aspect of communication are the questions posed by the teacher. Among these, particularly useful are inquiry questions that admit a range of legitimate responses, some of which may be rather interesting. Bishop and Goffree (1986) discuss the process of negotiation of mathematical meanings, leading students to make new connections among mathematics ideas, and leading the teacher to recognize their sometimes unforeseen points of view. In addition, Franke, Kazemi, and Battey (2007) stress the importance of processes that support students' language development, like revoicing, saying the same idea in a different way, usually closer to formal mathematical language.

The nature of tasks and classroom communication processes are key features of practices and characterize the mathematics teaching approach. One may say that the classroom seeking to provide students opportunities to solve tasks for which they do not have an immediate solution method follows an "exploratory approach" (Ponte, 2005; Ruthven, 1989). In this approach, the students work on tasks that involve challenging features often framed in contextualized situations, usually in pairs or small groups. The students are called to build or deepen their understanding of concepts, representations, procedures, and mathematical connections as they play an active role in interpreting the tasks proposed, in representing the information given, and in designing and implementing solving strategies, which they are called to present and justify to their colleagues.

Whole class discussions are a particular form of classroom work used in exploratory teaching that is attracting increasing attention of research in mathematics education. The teacher is called upon to prepare the discussion, seeking to make the best use of the work previously carried out by the students and the available class time. In this regard, Stein, Engle, Smith and Hughes (2008) underline the importance of the teacher anticipating the way students may think, monitor their work, collect the necessary information, select the issues to stress during the discussion, sequence the students' interventions, and establish connections among different solutions during the discussion. Such preparation is an important support to lead the discussion, but one must note that, besides the establishment of connections, a fruitful discussion involves many aspects that cannot be foreseen beforehand, and that the teacher needs to be prepared to face. As Sherin (2002) shows, this involves the need to balance aspects related to mathematics knowledge, requiring filtering ideas and focusing students' attention on fundamental mathematical notions, and also attention to aspects related to student's participation in the discussion. Schoenfeld (2014) considers whole class discussions as key kind of activity structure and led a project to develop a set of rubrics to study their quality (see http://map.mathshell.org/materials/background.php). Salient aspects of these rubrics include teachers' provision of classroom activities that support meaningful connections between procedures, concepts and contexts, provide opportunities for students' engagement in key practices, and lead the students in explaining their solutions and reasoning, responding to and building on each other's ideas. An important aspect concerns the uses of assessment, in which the teacher solicits student thinking and then builds on promising beginnings or addresses difficulties and misunderstandings.

Seeking to identify highly productive discussion situations, Wood (1999) highlighted the potential of the exploration of disagreements between students, seeking that they justify their positions and encouraging the other students to join the discussion. In addition, Potari and Jaworski (2002) called attention for the moments in which the teacher challenges the students mathematically. Bartolini Bussi and Mariotti (2008) describe a recurrent pattern of four steps in mathematical discussions based on the use of artifacts: "ask to go back to the task; focalize on certain aspects of the use of the artifact; ask for a synthesis; synthesize" (pp. 778-779). Cengiz, Kline and Grant (2011) proposed a framework of analysis for teachers' actions in leading mathematical discussions that distinguishes three fundamental kinds of actions—leading students in presenting their methods (eliciting actions), supporting their conceptual understanding (supporting actions), and enlarging or deepening their thinking (extending actions).

Ponte, Mata-Pereira and Quaresma (2013) developed a framework to analyze discussions that establishes a distinction between the actions of the teacher directly related to the mathematical topics and processes and actions that are related to management of learning (Figure 1). Centering their attention on the actions related to mathematical aspects, they distinguish four fundamental kinds:

- *Inviting,* aiming at initiating a discussion;
- *Supporting/Guiding*, leading students in solving a task through questions or observations that point (explicitly or implicitly), the path that they may follow;
- *Informing*/*Suggesting*, introducing information, giving suggestions, presenting arguments or validating students' responses;
- *Challenging,* seeking that the students produce new representations, interpret a statement, establish connections, or formulate a reasoning or an evaluation.

These four kinds of actions may be found in lessons with very different characteristics, with different frequencies and roles that is interesting to study. In all these actions one recognizes fundamental aspects of mathematical processes such as representing (creating new representations or transforming given representations), interpreting (revoicing using different words or establishing connections with other concepts), reasoning (making inferences, using the provided information to arrive at new conclusions) and evaluating (making judgments about the aspects related to solving the task).

Research methodology

This study follows a qualitative and interpretive approach (Bogdan & Biklen, 1994) since we intend to study the teacher's practice in moments of whole class discussion of tasks, taking into account the meaning that she assigns to it. The teacher whose practice is object of study (the second author of this paper) had 6 years of experience at the time. In her class she sought to carry out exploratory teaching, following the guidelines of the mathematics curriculum.[2] The school where this study was held is located in a deprived rural area, 50km away from Lisbon, with a high unemployment rate and subject to an educational intervention plan from the Ministry of Education. The fifth grade class is composed of 22 students aged 10 to 12 years old. Students show little working habits and have different levels of involvement in school activities, but they are eager to accept new kinds of tasks. In a diagnostic lesson held to identify students' knowledge and difficulties, they showed trouble in using the language of fractions, for example, saying "second part" to indicate "a half", and also evidenced some misconceptions in tasks involving decimal numbers. However, relying on the notion of division, they were able to solve simple questions involving unit fractions as operators.

[2] This study is a secondary analysis made collaboratively by both authors on the video records and transcripts of the classes taught by the second author as a research on her own practice for a master degree carried out under the supervision of the first author.

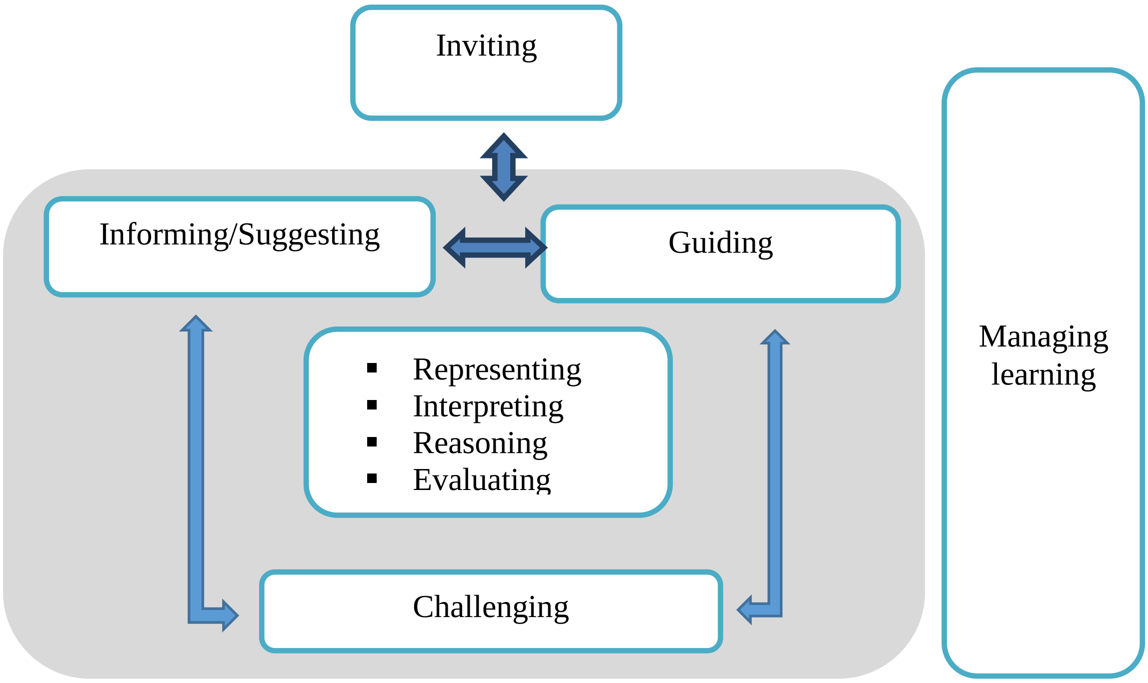

Figure 1. A framework to analyse teachers' actions (from Ponte et al., 2013).

The Portuguese curriculum indicated that students had to learn rational numbers using both fraction and decimal representations and working with the different meanings of rational numbers, most especially part-whole, quotient and measure. However, since this curriculum was just being implemented in primary school at the time, the students in this class worked mostly with decimal representations and only met fractions as operators (with simple unit fractions). Striving to follow the orientations of this curriculum, in this teaching unit, the teacher sought to work with the different meanings of rational numbers and aimed to promote students' flexibility in converting among decimal, fraction, pictorial, percent, and verbal representations. In her view, students should work on tasks with open features, formulated in both every day and mathematical contexts. She valued dialogic communication processes emphasizing inquiry questions. She also supported students' intuitive and informal strategies as well as their use of prior knowledge. The pictorial and decimal representations of rational numbers known by students were used as a basis to learn about fractions. In learning new representations, students were encouraged to keep using the former representations, choosing the most effective one for each problem. In all situations that we present, the teacher's general plan was to handle tasks in three phases: (i) presentation by the teacher and collective interpretation; (ii) exploration by the students in groups or pairs; and (iii) collective discussions

(Christiansen & Walther, 1986). Collective moments were highly valued assuming that they provide opportunities for negotiation of mathematical meanings and construction of new knowledge. We term such teaching, framed by tasks with open features and inquiry questions, as “exploratory classroom”.

The data was collected in the classes that addressed the topic of rational numbers. The classes were recorded on video, and the collective discussions were transcribed in full. We analyse the teacher’s practice in four tasks that illustrate different issues related to rational numbers such as learning about representations, concepts, procedures, connections, and reasoning. Data analysis began by identifying the threads in the discussion of the solution of each task, coding the teacher’s actions according to the categories shown in Figure 1. An inter-rater reliability study involving an independent researcher showed that this coding had 84% of agreement with the authors’ own coding,[3] which was found satisfactory. Then, we sought to establish relationships between these actions and significant events as regards students’ work with mathematical representations, developing concepts, procedures, connections, and reasoning (generalizations and justifications).

Teachers’ practice to foster mathematics learning

Next, we present episodes from the work on four tasks described in Quaresma (2010). These episodes represent different features of teachers’ practice leading whole class mathematical discussions aiming to promote students’ learning about representations, concepts, procedures, and connections, and to develop reasoning, as they work in pairs or small groups.

Dealing with representations

Learning about the different representations of rational numbers and their relationships is an important aim in the study of this topic. Task 1 (Figure 2), was proposed in the first lesson, in order to introduce the formal representations of rational numbers as fractions and percent and also the associated oral language. The second question aimed to get students started in comparing rational numbers. In these questions the most challenging feature was the interpretation of the key terms “represent” and “compare”.

[3] All instances of the teacher’s actions were coded independently by one of the authors and an external researcher to find the percent of agreement.

1. Find three strips of paper geometrically equal. Fold them in equal parts: a) The first in two; b) The second in four; c) The third in eight. After you fold each strip, represent in different ways the parts that you got.

2. Compare the three parts obtained by folding the strips. Record your findings.

Figure 2. Task 1, *Folds and more folds*.

*Learning the meaning of "representing" and "comparing"***.** At first sight, the task may seem very easy for students at this age level. In fact, that was not the case, as questions 1 and 2 proved to be quite demanding in terms of interpretation. In question 1, the students, working in groups of four or five, easily folded the strips as indicated in the first part of the question, but showed great difficulty in interpreting what is meant by "represent in different ways". As they did not know what to do, they started asking the teacher for help. The teacher recognized that the situation required a collective negotiation of meaning about what "to represent" means in mathematics. She used an example, drawing on the board the strip split in half, and asked students to tell which part was painted. Using oral language, all students said "half the strip". The teacher insisted asking for another way to represent that, and some students indicated the decimal representation "0.5". The teacher asked yet for other forms of representation and two students indicated the fraction "one of two", which the teacher revoiced as "a half". Finally, as students did not indicate any further representation, she asked, "and if I wanted to represent a percent? Would that be possible?" Here most students immediately said "it is 50%". In this way, the teacher began with a suggesting action, as she represented the strip split in half, challenged the students to provide other representations, and guided them with a more direct question about the possibility of representing the result as a percent.

In introducing question 2, the teacher carried out another negotiation of meaning as she realized that the students did not understand what to "compare the three parts obtained" is. Again, the students did not know what to do. In this case, the teacher began by showing the first two strips (representing $\frac{1}{2}$ and $\frac{1}{4}$) and asked the students to compare them. Some immediately indicated that $\frac{1}{4}$ "is half" of $\frac{1}{2}$. The teacher's question, asking the students just to compare two strips (instead of three), reduced the complexity, but still comported some degree of challenge in formulating a mathematical relationship based on concrete objects. This made the interpretation possible for the students and they went on working in the task.

Question 1 begins in a straightforward way by explicitly asking students to do several folds, but the request to represent parts of the strips "in different ways" is open to multiple interpretations. Question 2 also raises the question of what to "compare" parts is. Thus, the need to interpret and transform the questions proposed in the task in more explicit mathematical questions provided opportunities for students' exploratory activity. As the students had no clear meaning for the terms involved, these negotiations of meaning, undertaken in whole class discussions in the introductory stage of the work, involving suggesting, challenging, and guiding actions by the teacher were essential to adjust the level of challenge so that the task could be understood by the students. This episode shows how students may be asked to provide representations for describing mathematical objects and relationships. It also shows the kind of difficulties that students often experience as they try to interpret a task, and how that may be handled by the teacher in a collective discussion. Finally, it shows how the teacher seized the opportunity to support students' learning of a correct mathematical language to speak about fractions.

Supporting the use of mathematical language. In question 2 of Task 1, all students' groups established relationships among the parts using verbal language. Some groups were able to compare all strips, establishing simple relationships (as in Figure 3). However, the teacher wanted the students to use more elements of mathematical language in their responses, which did not happen.

- The 1st is the double of the 2nd	- The 2nd is the half of the 1st
- The 3rd is the half of the 2nd	- The 2nd is the double of the 3rd
- The 1st is the fourfold of the 3rd	- The 3rd is the fourth part of the 1st

Figure 3. Answer of Mariana's group, question 2.

André and his colleagues, besides simple relationships, such as "half" and "double", established more complex relationships such as "fourfold" (based on "double of the double") and "fourth part" ("fourth half" as one of them said, to mean "half of the half"). The teacher seized this opportunity to foster the development of the students' mathematical language. She challenged the students by asking them to indicate one of the relationships that they found using mathematical language:

Daniel: The second is half of the first.

Teacher: How can I write that using numbers? How do I do one half?

André: Dividing by 2.

Rui: One of four is equal to one half divided by 2.

Based on the pictorial representations of the strips, the students found several relationships among $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{8}$. They were able to compare the three fractions, showing understanding of how some rational numbers are related to each other and expressing these relationships in oral language. However, they showed difficulties in speaking about fractions, which was already known from the diagnostic lesson. These difficulties in using mathematical language were natural since this was the first lesson on this topic. The teacher addressed this by revoicing students' interventions, saying "one half" as an alternative for "one over two", and "one-eighth" as another way for "one dash eight". Such revoicing clarifies what the student says, thus strengthening the access of the class to mathematical ideas. In addition, it yields the student the status of a person who says something mathematically important, strengthening the notion that students are also mathematical authorities. Realizing the difficulties that the students had in expressing themselves in mathematical terms, the teacher decided to help them in developing their language, challenging them to write these relationships in a more formal way. This challenge was supported by a guiding action from the teacher, taking advantage of the opportunity provided by the students' solutions.

This task aimed at having students finding and representing relationships among quantities, and again allowed for a variety of responses creating a setting for a productive mathematical discussion. The task also aimed at students using mathematical language to express those relationships and the students' responses led the teacher to notice the need for further work on this issue. This episode shows how collective discussions may be useful to promote students' ability to express mathematical ideas in oral and written form.

Fostering the development of concepts

The development of concepts is an essential aspect of mathematics learning. Task 2 (Figure 4) was proposed in lesson 6 aiming to lead students to use rational numbers in the meanings part-whole and operator and to introduce equivalence of fractions. It is a contextualized situation, in which the information is given in verbal representation and the answer is sought as a fraction. Question 1 asks to use fractions as operators to construct the part. Question 2 asks to represent nine objects out of twelve by a fraction, making the appearance of equivalent fractions possible. The challenging feature of this task is that it was the first time that students were dealing with the operator meaning with discrete quantities.

1. Carlos collects stickers. When he had 6 stickers he lost two sixths of the stickers. How many stickers did he lose? You may solve this using words, drawings, manipulatives or computations.

2. Carlos' friend had 12 stickers and gave 9 to Carlos. What fraction of his 12 stickers did he give to Carlos? You may solve this using words, drawings, manipulatives or computations.

Figure 4. Task 2, *Collecting stickers*.

The students did not show difficulties in solving question 1, making the correspondence between the denominator of the fraction and the total number of stickers:

Nuno: If he had six stickers and lost $\frac{2}{6}$, he lost two stickers.

Teacher: Explain then how you thought.

Nuno: So $\frac{2}{6}$ are two of six. If he had six, he lost two of six.

As the denominator of the operator corresponds to the total number of stickers, the students solved this question based on the part-whole meaning. We note the teacher's guiding question ("Explain then…"), formulated with the purpose of leading Nuno to make his strategy explicit.

In question 2, most students used the simpler fraction, based on the statement of the question, indicating "nine of twelve", which the teacher revoiced as "nine twelfths". Miguel was able to go even further, realizing that $\frac{9}{12}$ also may be represented by the equivalent fraction $\frac{3}{4}$, as he wrote on his notebook ("He yielded $\frac{9}{12}$ or $\frac{3}{4}$ of his stickers to Carlos"):

Miguel: Or else it could be 3 of 4. It is the same.

Teacher: Explain that to those who do not understand it.

. . .

Miguel: It is the same as if we made 3+3+3+3. It is 12 . . . If we made this way, he would give 9 of that… It is 3 of the 4 sets of 3. We took 9 but then we still had 3 stickers left. It is as if it was 3 of 4.

Teacher: So this (3 stickers) represents which part of the whole?

Class: One fourth.

Leonor: Yes, it is. If 3 stickers are $\frac{1}{4}$, 6 stickers will be $\frac{2}{4}$…

Teacher: All this (6 stickers) $\frac{2}{4}$.

Leonor: And what we got, the 9 stickers, represent $\frac{3}{4}$.

Teacher: And all $\frac{3}{4}$ …

Leonor: The 9 that he gave are $\frac{3}{4}$ of 12 stickers.

Miguel recognized by himself that the fraction $\frac{9}{12}$ was equivalent to the simpler fraction $\frac{3}{4}$. However, he had trouble in explaining the way he had thought, and many of his colleagues did not understand what he was saying, showing difficulty in understanding composite units.

The aims of the task were achieved as a student spontaneously introduced equivalent fractions, thus allowing for a discussion on fraction equivalence. The statement of the task invited the students to use a variety of representations, leading to a reflection about the suitability of each representation. The teacher realized that as an opportunity to address the concept of equivalence of fractions and decided to explore the situation in depth. In this case, there was no need to make an explicit challenge – just by using guiding and supporting actions, the teacher led the student to make his idea more explicit using the language of rational numbers and another student, Leonor, understood his discovery and explained it in her own words.

Developing procedures

Procedures enable a quick solution of a whole class of problems and constitute another important aspect of mathematics learning. Task 3 (Figure 5), presented in a purely mathematical context, asks students to order four numbers represented in different ways: fraction, decimal and percent. It was proposed to the students in lesson 7 and presents an opportunity to formulate a procedure for ordering rational numbers given in different representations. The challenging aspect of this task was that students did not have a ready-made strategy to solve it.

Write in increasing order the following numbers: $\frac{\mathbf{1}}{\mathbf{4}}$; $\frac{\mathbf{7}}{\mathbf{10}}$; 26%; 0.267.

Figure 5. Task 3, *Order four rational numbers in different representations.*

Miguel presented his solution on the board: $\frac{1}{4}$; 0.267; 26%; $\frac{7}{10}$. He recognized that $\frac{1}{4}$ is the smaller number (still showing difficulty in saying it verbally) but he incorrectly ordered 26% and 0.267. This led Leonor to intervene, pointing this mistake. However, her argument was framed in the percent representation, and the teacher was faced with the problem how to conduct a discussion so that all students could understand. She rephrased the student's intervention in a slightly more general way ("if you transform this in percent") to help Leonor to move on in explaining her reasoning (guiding action):

Leonor: I think it is wrong. $\frac{\mathbf{1}}{\mathbf{4}}$ is right, but the other would yield 26.7%.

Teacher: If you transform this in percent…

Leonor: It would yield 26.7%.

Teacher: And so, do you agree [that it is not right]?

Students: Yes.

Leonor: We have to interchange 0.267 and 26%.

Then, with a suggesting action, the teacher introduced another possibility to solve this question, which would be to transform the number given in percent into a decimal:

Teacher: But if we could also transform 26% in a decimal, what would it be?

Leonor: 0.260.

Teacher: Then comparing the percent or the decimals we reach the same conclusion. So, and $\frac{\mathbf{7}}{\mathbf{10}}$, it goes there why?

Student: It is as if it was 0.700.

Teacher: And in percent?

Leonor: 70%.

Taking advantage of an incorrect solution provided by one student (Miguel) and a correct but not unique solution by another student (Leonor), the teacher led the students in the class to recognize that they could easily order the numbers transforming all of them into the decimal representation, and showing that this would be a procedure to use in ordering questions. Finally, she completed the solution of the question asking about where to put $\frac{7}{10}$, making sure that the students were able to use the two representations (decimals and percent) that had already been used in this task.

In this situation the teacher faced the issue of dealing with an incorrect response from a student, which another student immediately indicated as being wrong. However, the way this second student expressed herself did not make the reasoning completely clear, and the teacher made several interventions with guiding and suggesting actions, leading the whole class to figure out a procedure to compare rational numbers given in different representations. The task presented an opportunity to construct a procedure in a natural way and this was explored by the teacher. As in previous episodes, the variety of answers provided by students were used as starting points for a mathematical discussion. The teacher aimed to help students to arrive to this procedure by themselves, at the same time that she dealt with the issue of students' difficulty in using acceptable mathematical language that everyone could understand.

Fostering the development of connections and reasoning

Task 4 (Figure 6) was also proposed in lesson 6 seeking the development of connections and reasoning. It involves the quotient meaning, in a contextualized situation with continuous quantities. Its aim was to promote the notion of fair sharing, fostering the quotient meaning of rational numbers, to compare a quantity with another taken as the unit, and to compare rational numbers in different representations. The information is given verbally and there is no indication about the representation to be used in the answers. The challenging features of this task is to make sense of the quotient meaning of rational numbers (three pizzas shared by four/eight friends) and to find relationships between the two situations (four/eight friends).

1) Four friends went to a restaurant and ordered three pizzas. They divided the three pizzas equally. 1.1) Which part of the pizzas did each friend eat? 1.2) Did each friend eat more or less that one pizza?

2) If eight friends were to divide three pizzas, which part of pizza would each friend eat?

3) In which of the previous cases, four friends (question 1) or eight friends (question 2), did each friend eat more pizza? Explain your reasoning.

Figure 6. Task 4, *Sharing pizzas.*

*Working in contexts related to everyday life***.** Question 1 asks students to share three pizzas for four friends equitably, and then compare the amount that each friend got with the unit. The students began working on this task in a very enthusiastic way, immediately drawing three pizzas. However, many students were not able to share the pizzas among the four friends, which led the teacher to suggest that they could give names to the slices, that is, simulate the actual process of equal sharing among the four friends. Most students were then able to share the three pizzas among the four friends, using the pictorial representation of the pizzas (see an example in Figure 7).

Figure 7. Answer of Rui and Leonardo to question 1.

While the students were doing the task in pairs, the teacher found out that all of them represented the situation both pictorially and by fractions. This was a surprise for her. So, she guided a student in presenting more details of his solution expecting that some talk about fractions would emerge:

Teacher: And then, did you do anything else?

Leonardo: Yes, we made computations.

Teacher: What computations did you make?

Leonardo: We made one over four…

Teacher: One fourth…

Leonardo: Times three is equal to three fourths.

The teacher found the solution of Leonardo and Rui quite interesting, since the students presented the part that each friend ate as the product $\frac{1}{4} \times 3$. Multiplication of a fraction by a whole number had never been addressed in class up to that point. So, the students had made a generalization based on their intuitive knowledge on whole numbers.

The teacher found this remarkable, but decided to pick up on it later. For the time being, as some students were not convinced with this solution, the teacher decided to continue the discussion challenging Leonor to establish connections between her solution and that of her colleagues:

> Leonor: We counted at once. Three from A, three from B… Each one ate 3 parts, so they ate $\frac{3}{4}$.
>
> Teacher: So, you immediately understood that each slice corresponded to the size…
>
> Students: One fourth.
>
> Amélia: Oh, teacher, but if each one ate 3 parts, it was then $\frac{3}{4}$.

Amélia and Leonor, who had worked together, began directly from the picture, also showing to understand that each part represents $\frac{1}{4}$ of the pizza and, therefore, if each friend ate 3 slices, each one ate $\frac{3}{4}$ of a pizza.

In this episode, the teacher began by helping the students to solve the problem in the pictorial representation, using this representation in a simulated way as it would be done in real life. This task proved to be challenging for the students because of the nature of the quantities involved and allowed for interesting solving strategies to appear, including the formulation of generalizations and justifications. It also allowed the introduction of new mathematical ideas, such as the multiplication of a whole number by a fraction.

An important teacher decision was to take the opportunity provided by the fact that all students also used fractions to represent the situation, and she guided them in presenting and interpreting their solutions using this representation. Some students seemed not to be convinced by the solution presented by one of them (Leonardo), so the teacher implicitly challenged another student (Leonor) to present her solution and to make connections with the previous one, and challenged her to justify her answers as well. Therefore, the teacher in this episode had an important role in stressing connections to the context of the task to support students' learning.

*Encouraging students to reason using representations and contextual elements***.** Question 1.2 asks students to compare the part that each friend ate with the unit. Some students used practical and intuitive everyday knowledge to answer the question. Responding to the teacher's invitation, Leonor presented a brief answer with no justification. With an inquiry question, the teacher challenged the student to justify her answer, which Leonor did based on her mathematical knowledge about fractions. Another student, André, provided a second justification:

> Teacher: Each friend ate more than one pizza or less than one pizza?
>
> Students: Less.
>
> Teacher: Less… Why, Leonor?
>
> Leonor: Because they only ate $\frac{\mathbf{3}}{\mathbf{4}}$ and a whole pizza has $\frac{\mathbf{4}}{\mathbf{4}}$.
>
> André: Oh, teacher, it is not just that. We see that at once because there are 3 pizzas and they are 4; so there is no way they can eat more than one unit.

Question 3 of task 4 required comparing the parts obtained in questions 1 and 2 ($\frac{3}{4}$ and $\frac{3}{8}$) that is, to compare fractions with equal numerators and different denominators. During the discussion, the students readily indicated the relationship between the size of the portion corresponding to every friend in both cases ($\frac{3}{4}$ and $\frac{3}{8}$).

> Teacher: What is the difference between what each one eats in the first case and in the second case? What has changed?
>
> Amélia: It becomes sliced into more parts.
>
> Teacher: What happened to each part?
>
> Students: It becomes smaller. It becomes one half.
>
> Leonor: Yes, it is, $\frac{\mathbf{3}}{\mathbf{8}}$ is the half of $\frac{\mathbf{3}}{\mathbf{4}}$!
>
> Teacher: That means that each one went on to eat which part of what they ate in the first case?
>
> Students: The half.
>
> Teacher: Yes, since we duplicated the number of friends, each had to share a slice with another.

As Amélia said that "it becomes sliced into more parts", the teacher decided to further challenge the students to provide a more precise relationship. This challenge was successful as some students immediately said that $\frac{3}{8}$ is half of $\frac{3}{4}$, explaining that each part is half of the previous one. Rui wrote that in question 1 each friend eats more: "in question

1 it is one fourth and in question 2 it is half of one fourth.” The students figured out the generalization that doubling the number of people implied that each one would eat a half of the former piece. We note that this situation presents a meaningful context for the students that led them to easily recognize that $\frac{3}{8}$ is half of $\frac{3}{4}$, which would probably be more difficult to recognize if they had worked only with the numbers with no reference to the everyday context.

Other students also used the logic of the situation to compare the two fractions:

> Teacher: In which of the former groups did each friend eat more pizza?
>
> Nuno: I think that it was in question 1, because in [question] 2 we had to divide the pizzas by 8 people and in [question] 1 we only had 4 people.
>
> André: That is right, we had less people to distribute.
>
> Teacher: And the amount of pizza is always the same, right?
>
> Leonor: Yes, we also thought in the same way, but we said the slices would be smaller that way.
>
> Amélia: We may conclude that they ate the same number of slices, but as we give them to less people [in question 1] the slices are larger.

This task provided opportunities for making justifications and generalizations that were explored by the teacher in a fruitful way. The fact that a student (Leonor) provided a correct answer with no justification, prompted the teacher to make a challenging action to lead her in explaining her reasoning, by justifying her statement. Amélia’s answer also led the teacher to challenge her to provide a better one. Then, the teacher guided the students in expressing their reasoning about this question in a subtle way, since she opened the door for students’ answers without giving too many clues. She encouraged the students to use informal procedures as well as practical knowledge to compare and establish relationships among fractions and to support the development of students’ reasoning.

In the last dialogue, as in several former ones, the intervention of a student (Nuno) is followed by the intervention of another student (André). This happens because the students understood that they could express their agreement or disagreement with the interventions of their colleagues. This represents a shift away from I-R-E sequences and is also a shift in norms. That is, the expectation that reasoning and explanations may can come both from students and the teacher. In this way mathematics is no longer just carrying out procedures, involving sense making and argumentation.

Discussion and conclusion

The teacher's activity sought to create conditions for students' learning about rational number concepts, representations, and procedures, and to develop their reasoning through exploratory work based on tasks with some degree of challenge in an environment of dialogic communication. Exploratory teaching practice was framed by the teacher's selection and presentation of tasks to the students—tasks that they did not know how to solve in an immediate way, but in which they could design a strategy to come up with a solution. That is, in an exploratory approach, the tasks involve an element of challenge that needs to be in the right measure. The fact that the students do not know how to respond may motivate them to find a strategy, unless they feel that the question is too hard for them. The mathematical knowledge required to solve these tasks must be within the students' reach. In these episodes, the tasks proposed to the students provided fruitful situations to learn and relate mathematical representations and to discuss their suitability in a given situation, to develop new mathematical concepts, connections, and procedures, as well as to develop students' mathematical language and mathematical reasoning. That is, based on their own exploratory work, the students learned about representing and comparing rational numbers, fair sharing, reconstructing the unit, equivalence of fractions, and how to order rational numbers given in different representations. They also had opportunities for generalizing and justifying.

A second key element that frames teaching practice is the way the teacher conducts classroom communication. Such communication is led by the teacher, and may include further elements of challenge, inviting students to explain their solutions, to make connections, to make conjectures, and to justify assertions. In order to achieve this, inquiry questions that require students' interpretation and may elicit different responses are most useful. In our study, some of the challenges the teacher posed were quite explicit, as when she asked a student to provide a justification (with "why" questions). Other challenges are more subtle, as they orient students' thinking in a given direction, but still leave significant room for students' interpretation and sense making. Challenging students is an important element of this teaching approach but in many cases the teacher needed to support the students, providing direction and promoting their confidence, mostly with guiding questions. This concurs with the perspective that tasks must be challenging, but sometimes this needs to be adjusted in classroom communication so that they fit within what students can understand.

Challenges may involve the establishment of connections between mathematical concepts of properties or between the context and the conditions of the problem or may be related to the construction, selection or coordination of representations. Challenges may also be related to promoting the reasoning processes of generalizing and justifying. The importance of challenging students is highlighted by Potari and Jaworski (2002) and Schoenfeld (2014) and frames many current curriculum documents such as NCTM (2014). This study shows how challenging elements may relate to different features of tasks and may be further nurtured during classroom communication, although set in varying degrees and supported by inviting and guiding actions from the teacher.

In this study we gave special attention to three dimensions that Schoenfeld (2014) uses to characterize the robustness of a learning environment, the cognitive demand (or level of challenge), the mathematical focus, and the discourse promoting students' agency, authority, and identity. Similarly to the higher level rubrics that this author uses to characterize whole class activities (http://map.mathshell.org/materials/background.php), in the episodes presented in this paper we see students engaging in classroom activities supporting connections between concepts, contexts and procedures and in representing, generalizing and justifying, struggling to cope with challenging questions, explaining their ideas and reasoning and responding to those of their colleagues.

The episodes presented in this study show how the teacher faced several problems and noticed several learning opportunities. For instance, when the students had difficulty in interpreting the statements of some tasks, she used that as an opportunity to make negotiations of meaning as with the notions "to represent" and "to compare". The teacher used wrong answers as the starting point to have students analyzing mistakes, and providing and justifying alternative solutions. The teacher used further unexpected strategies from students as opportunities to introduce new concepts, such as the equivalence of fractions, and to establish connections among different representations. Students' difficulties in expressing themselves prompted the teacher to make guiding actions, so that they could make themselves understood by their colleagues. Incorrect use of language led the teacher to revoice the students' talk in more correct mathematical language.

The focus on teachers' actions as inviting, informing/suggesting, guiding, and challenging enables an analysis of classes with different mathematical agendas, including providing students with opportunities for learning about representations, concepts, procedures, and connections, and to develop reasoning. Bartolini Bussi and Mariotti (2008) suggest a recurrent pattern of inviting students to present their ideas, focalizing some issues, asking for a synthesis, and synthesizing. In the episodes in our study there are also segments of discourse beginning with inviting actions and ending with informing/suggesting actions as a way of synthetizing. However, guiding, challenging, and informing/suggesting actions do not appear in a single pattern. Instead, they show up in different combinations and with different emphasis, depending on the opportunities and difficulties expressed by the students in the classroom. Challenging actions were particularly noticeable in the beginning of segments in which the teacher asked the students about different representations or prompted the establishment of connections, generalizations and justifications. In counterpart, guiding and informing/suggesting actions are most prominent in segments involving the introduction of new concepts and procedures.

In summary, this study suggests that, within an exploratory approach, some degree of challenge fosters fruitful learning situations. However, such situations often require some degree of preparation and of follow-up with guiding and even informing/suggesting actions in order that the students have an opportunity to learn what was set in the teacher's agenda. The tuning of the degree of challenge and the combination of different teacher's actions at different school levels, curriculum topics, teaching approaches, and activity structures are interesting issues to pursue in further studies.

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